Graph Algorithms: Depth-First Search (DFS)

Lecture 14A

Danqi Chen (Princeton University)

AddisCoder 2026

August 13, 2026

About me

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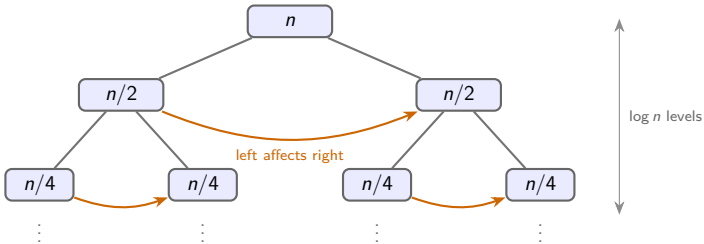
- I started learning **programming and algorithms** in high school — and was a competitive programmer, many many years ago
- My **first time in Ethiopia** — I hope to come back often!

Fun fact: CDQ's divide and conquer

An **offline** divide-and-conquer algorithm for problems where **updates** and **queries** are interleaved over time.



Learn more:

robert1003.github.io/2020/01/31/cdq-divide-and-conquer.html

codeforces.com/blog/entry/150934

My lecture plan

Thursday	Morning	**14A**	**Depth-First Search (DFS)** ← we are here
	Afternoon	**14B**	Breadth-First Search (BFS)
Friday	Morning	**15A**	Quiz
	Afternoon	**15B**	Language Models

Yesterday: graphs in code

Recall that a graph G is made up of **vertices** V and **edges** E.

```
import networkx as nx

G = nx.Graph()
G.add_edges_from([
    ("a", "b"), ("b", "c"),
    ("d", "b"), ("d", "c")])
nx.draw(G, with_labels=True,
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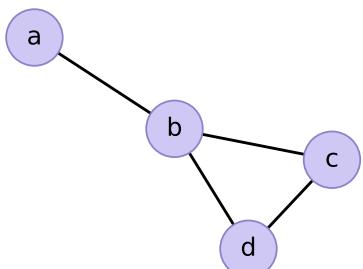

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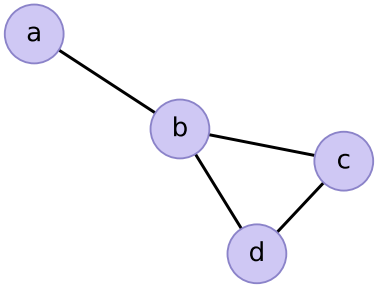


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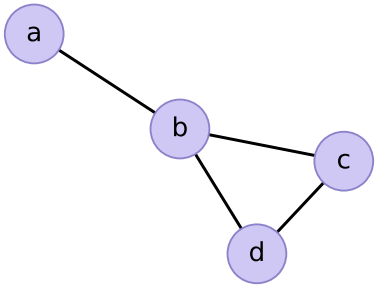


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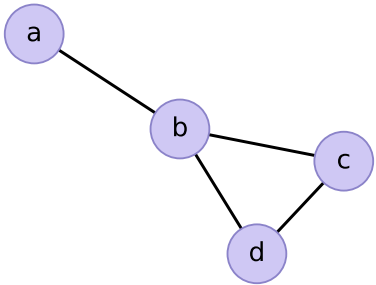


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- An **adjacency dictionary**: each vertex maps to its **neighbors**
- **Undirected**, so it is symmetric: `"a"` lists `"b"`, and `"b"` lists `"a"`

Directed graphs

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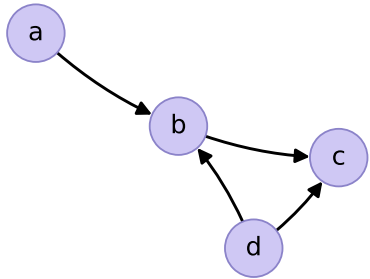

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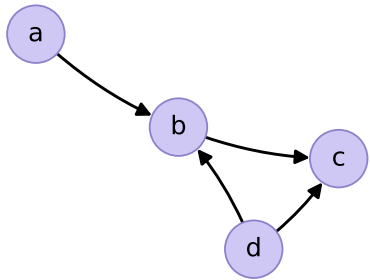

- `nx.DiGraph()` instead of `nx.Graph()` — same edges, now one-way
- No longer symmetric: `"a"` lists `"b"`, but `"b"` does **not** list `"a"`

Reachability

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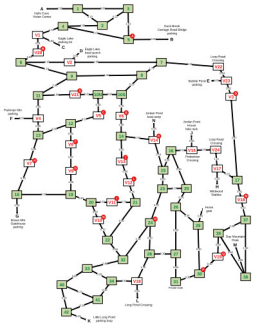

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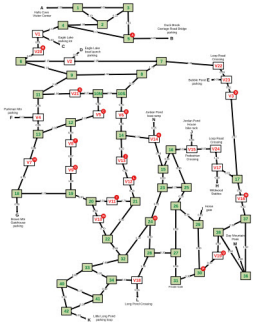

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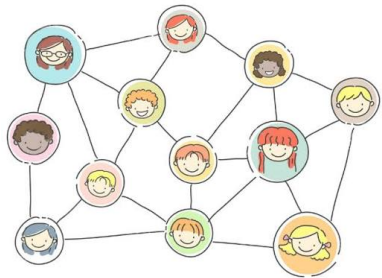
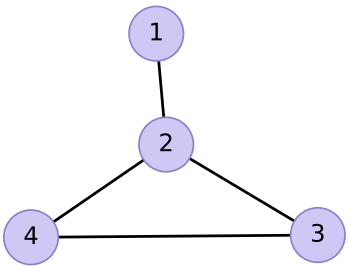


Are two people connected through friends of friends?

Reachability, precisely

- Vertex j is **reachable** from vertex i if there is a **path** from i to j



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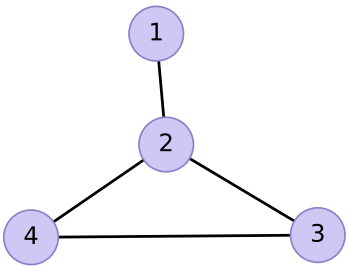

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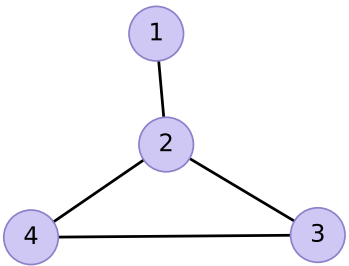

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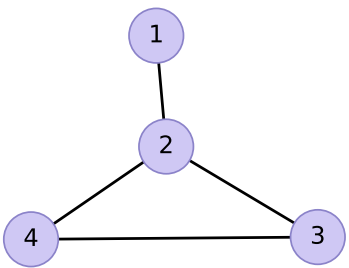

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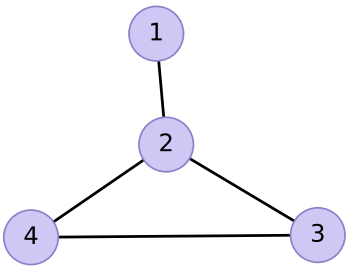

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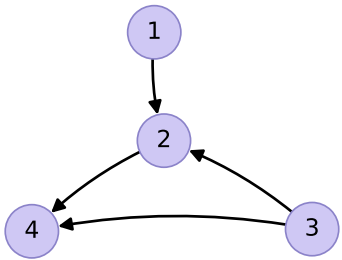

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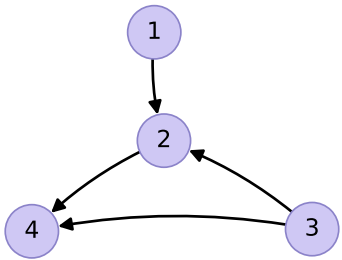

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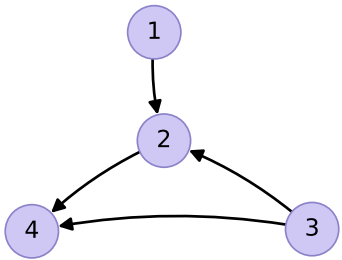

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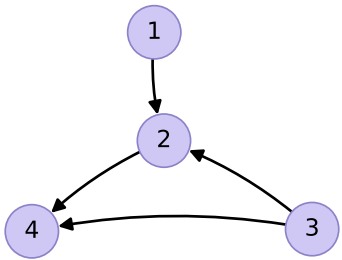

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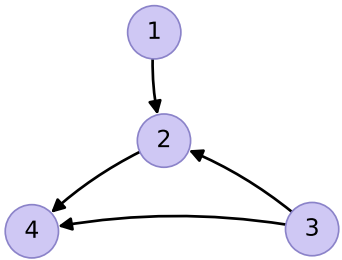

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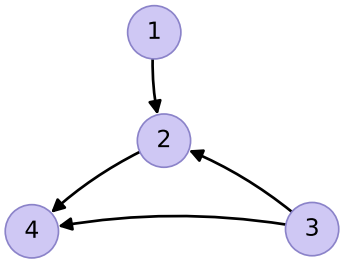

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But how do you know if vertex j is reachable from vertex i?

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- A definition in terms of itself... that's **recursion**!

Let's turn that idea into code

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Our test graph

```
g_connected = {1: [2, 3],
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               5: [3, 6],
               6: [5]}
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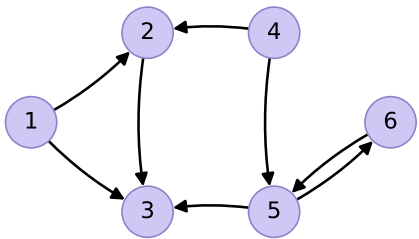

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Discuss: What will `connected(5, 2, g_connected)` return — `True`, `False`, or something else?

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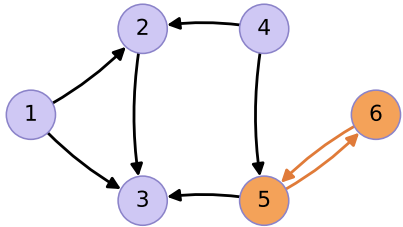
*The calls **never stop**. Python gives up:* `RecursionError`.

What went wrong?

- 5 and 6 point at **each other**



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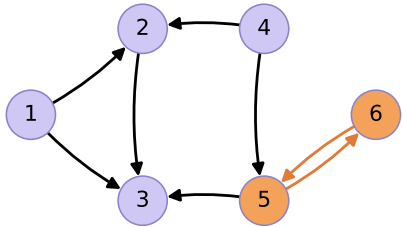

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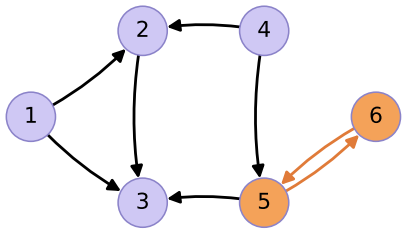

Discuss: How would **you** avoid walking in circles in a maze?

The fix: remember where you've been

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def connected(i, j, G, visited):
    if i == j:
        return True
    visited[i] = True    # leave a breadcrumb
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- Two new lines — that's the whole fix

Does it work?

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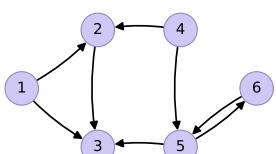

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- DFS is a **graph traversal** algorithm: it lets us move through every node of a graph in a structured way
- To do that, we have to **keep track of the nodes we have already seen** — so we know which ones are still left to visit

DFS, step by step

1. Start at a node

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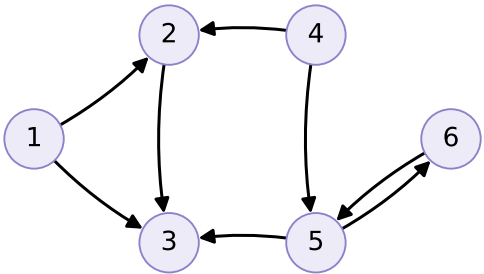
Go deep first. Back up only when you must.

Let's trace it by hand: DFS from 4

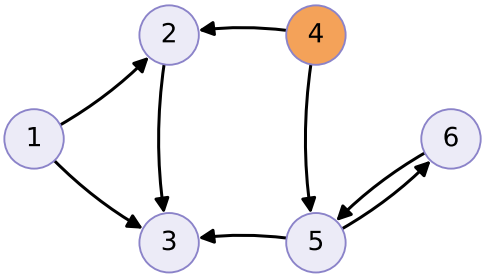


Now: start at 4

Call stack:

`visited: {}`

Let's trace it by hand: DFS from 4



Now: visit 4 — `G[4] = [2, 5]`

Call stack: `4`

`visited`: `{4}`

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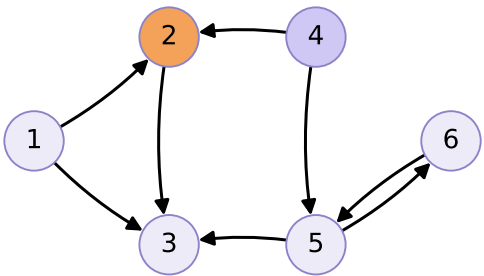

Now: visit 2 — `G[2] = [3]`

Call stack: `4, 2`

`visited`: `{4, 2}`

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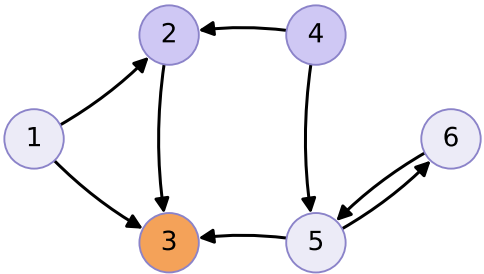
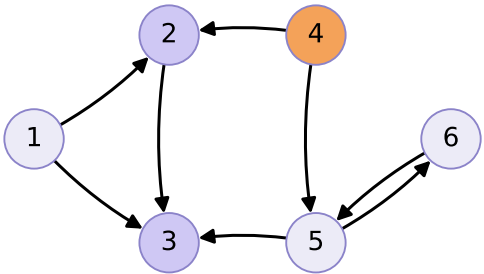


Now: visit 3 — `G[3] = []`

Call stack: `4, 2, 3`

`visited`: `{4, 2, 3}`

Let's trace it by hand: DFS from 4



Now: `G[3] = []` → backtrack

Call stack: `4`

`visited`: `{4, 2, 3}`

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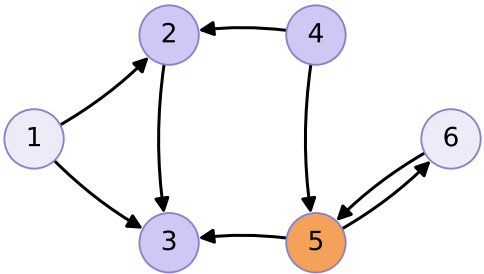
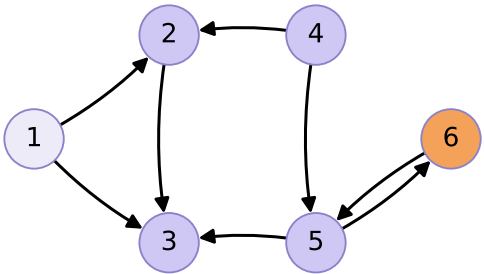


Now: visit 5 — `G[5] = [3, 6]`

Call stack: `4, 5`

`visited`: `{4, 2, 3, 5}`

Let's trace it by hand: DFS from 4



Now: 3 visited → visit 6

Call stack: `4, 5, 6`

`visited`: `{4, 2, 3, 5, 6}`

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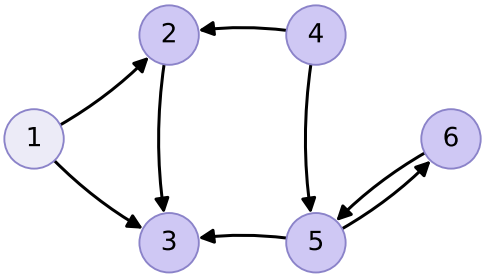
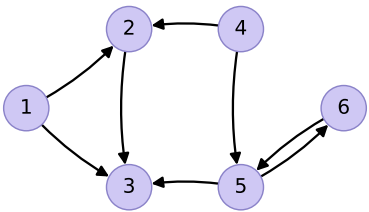


Now: 5 visited → done!

Call stack:

`visited:` {4, 2, 3, 5, 6}

What if we start at 1?



Discuss:

In what order does DFS visit the nodes?

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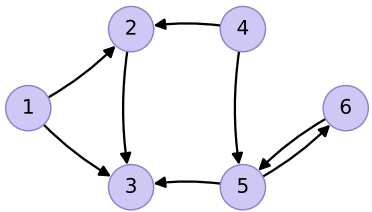

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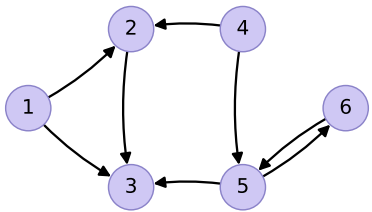

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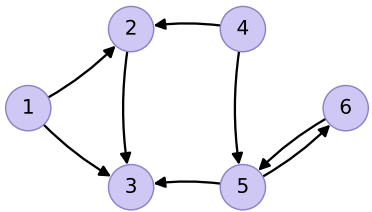

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4, 5, 6

Let's implement DFS

Converting graph edges to an adjacency dictionary

```
# convert graph edges to adjacency dictionary
def convert_adjacency(edges):
    res = {}
    for i in edges:
        res[i[0]] = []
        res[i[1]] = []
    for i in edges:
        res[i[0]] += [i[1]]
    return res
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```
g = [[1, 2], [2, 3], [4, 2], [1, 3], [5, 3], [4, 5], [5, 6], [6, 5]]
g_adj = convert_adjacency(g)
print(g_adj)
# {1: [2, 3], 2: [3], 3: [], 4: [2, 5], 5: [3, 6], 6: [5]}
```

Depth-first search in code

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def dfs(visited, graph, node):
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```

Six lines — the same shape as `connected`, but with no target `j` it ***visits everything*** *it can reach.*

Running it from every node

```
# g_adj = {1: [2, 3], 2: [3], 3: [],
#          4: [2, 5], 5: [3, 6], 6: [5]}

for i in g_adj.keys():
    visited = {}
    print("DFS from", i)
    dfs(visited, g_adj, i)
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2
3
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Discuss: How many times does DFS **visit** each vertex? How many times does it **look at** each edge?

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- Each **vertex**: visited once — `visited` makes sure of that
- Each **edge**: looked at once — when we scan its start's neighbor list
- Total work: $O(V + E)$ (V vertices, E edges)

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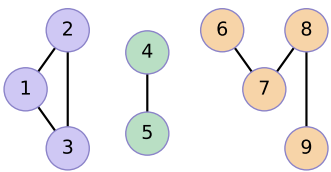

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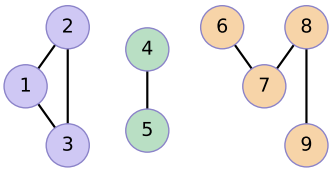

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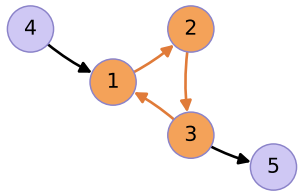
3. **Detecting cycles** in a directed graph



If DFS reaches a node still on the call stack — there's a cycle.

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This afternoon:

- A search that explores **level by level** instead
- Almost the same code — **one** data structure changes