Lab 5

Exercise 1: Find simple functions f (like n, or n^5, or $n \log_2 n$) for each of the functions below so that they are $\Theta(f)$. For the recurrences T below, assume $T(n) = 1$ for $n \leq 2$, and otherwise satisfies the recurrence given.

- n^3

  n^3

- $.5n^3$

  n^3

- $10n^7$

  n^7

- $\sum_{i=0}^{n} i^2$

  n^3. This sum is $O(n^3)$ since it is at most $n \cdot n^2 = n^3$. It is also $\Omega(n^3)$ since it is at least $(n/2) \cdot (n/2)^2 = n^3/8$.

- $\sum_{i=0}^{n} i^3$

  n^4. This sum is $O(n^4)$ since it is at most $n \cdot n^3 = n^4$. It is also $\Omega(n^4)$ since it is at least $(n/2) \cdot (n/2)^3 = n^4/16$.

- $\sum_{i=0}^{n} \sqrt{i}$

  $n^{3/2}$. This sum is $O(n^{3/2})$ since it is at most $n \cdot \sqrt{n} = n^{3/2}$. It is also $\Omega(n^{3/2})$ since it is at least $(n/2) \cdot \sqrt{n/2} = n^{3/2}/(2\sqrt{2})$.

- $\sum_{i=1}^{n} \sqrt{i} \cdot \log_2 i$

  $n^{3/2} \log_2 n$. This sum is $O(n^{3/2} \log_2 n)$ since it is at most $n \cdot \sqrt{n} \log_2 n = n^{3/2} \log_2 n$. It is also $\Omega(n^{3/2} \log_2 n)$ since it is at least $(n/2) \cdot \sqrt{n/2} \log_2(n/2) = n^{3/2}(\log_2 n)/(2\sqrt{2}) - n^{3/2}/(2\sqrt{2})$.

- $T(n) = T(n-1) + 5$

  n. We have $T(n) = 5 + 5 + 5 + \ldots + 5 + 1$, where the number of 5s appearing is $n - 2$, since $T(2) = 1$. So for $n \geq 2$, $T(n) = 5n - 10 + 1 = \Theta(n)$.

- $T(n) = T(n-2) + 2n$

  n^2. We have $T(n) = 2(n + n - 2 + n - 4 + \ldots + 4) + 1$. So, $T(n) \leq 2n^2/2 = O(n^2)$. Also, at least $n/4$ terms in the sum are $n/2$, so $T(n) \geq 2(n/4)(n/2) = \Omega(n^2)$. So, $T(n) = \Theta(n^2)$.

- $T(n) = T(\sqrt{n}) + 1$

$\log_2 \log_2 n$. The number of recursive steps in computing $T(n)$ before we get down to $n = 2$ is the k such that $(((n^{1/2})^{1/2})^{\cdots})^{1/2}$ (k times) is at most 2. Thus, we want $n^{1/2^k} \leq 2$. Rearranging gives $k \geq \log_2 \log_2 n$.

- $T(n) = 2T(n/2) + \log_2 n$

  n. If one draws the recursion tree for $T(n)$ as we did in class, then the bottom level (let's call it level 1) has $n/2$ leaves that each contribute 1 to $T(n)$. Going up the levels of the tree, we see that the kth level contributes $(n/2^k) \cdot k$ (there are $n/2^k$ nodes, each contributing k). Thus, $T(n)$ is at most $\sum_{k=1}^{\infty}(n/2^k) \cdot k = n \cdot \sum_{k=1}^{\infty} k/2^k = O(n)$. It is also $\Omega(n)$ since level 1 alone already contributes $n/2$.

Exercise 2: Recall the Fibonacci recurrence

$$\text{fib}(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 1 \\ \text{fib}(n-1) + \text{fib}(n-2) & \text{otherwise} \end{cases}.$$

Find a value c so that $\text{fib}(n) \leq c^n$. Prove that this is so using induction.

Solution: $c = (1 + \sqrt{5})/2$. For the base case with $n = 0$, we want $\text{fib}(0) \leq c^0$. This would be true for any $c > 0$ since $c^0 = \text{fib}(0) = 1$. It would also be true for $n = 1$ as long as $c \geq 1$. Now for the inductive case for $n > 1$, we assume $\text{fib}(j) \leq c^j$ for $j = 0, 1, \ldots, n-1$ and want to show that it's true for $j = n$.

Well, we have $\text{fib}(n) = \text{fib}(n-1) + \text{fib}(n-2) \leq c^{n-1} + c^{n-2}$. We want $c^{n-1} + c^{n-2} \geq c^n$ to make our inductive hypothesis go through. So, let's find a c so that $c^{n-1} + c^{n-2} = c^n$. Dividing through by c^{n-2} and rearranging, we have $c^2 - c - 1 = 0$. Solving this quadratic equation gives $c = (1 + \sqrt{5})/2$. Note that our simple recursive algorithm for computing $\text{fib}(n)$ took $\text{fib}(n)$ steps, so its running time is also $\Theta(((1 + \sqrt{5})/2)^n)$.

Exercise 3: Show by induction that every integer 2 or greater is a product of primes.

Solution: We show by induction the claim that every integer $n \geq 2$ is a product of primes.

- **Base case:** The claim is true for $n = 2$ since $2 = 2$.

- **Inductive step:** We suppose the claim is true for $0, 1, \ldots, n-1$ and want to show it for some $n \geq 3$. If n is prime, then we can write $n = n$ and are done. Otherwise, if n isn't prime, that means we can write $n = m \cdot r$ for some integers $1 < m, r < n$. By our inductive step since $m, r < n$, we can write m as a product of primes $m = p_1 \cdots p_x$ and r as a product of primes $r = q_1 \cdots q_y$, so we can write n as a product of primes as well $n = p_1 \cdots p_x q_1 \cdots q_y$.

Exercise 4: Suppose a country only has 3-cent and 5-cent coins. Show by induction that you can make change for any monetary value which is at least 8 cents.

Solution: Our claim is that we can make change for n cents as long as $n \geq 8$.

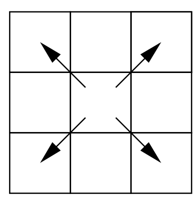
- **Base cases:** For $n = 8, 9$, or 10 we can make change as $8 = 3 + 5$, $9 = 3 + 3 + 3$, $10 = 5 + 5$.
- **Inductive step:** For $n \geq 11$ we can assume that it's true for $8, 9, \ldots, n - 1$. Then we can make change for n cents by making change for $n - 3$ cents then adding in a 3-cent coin.

Exercise 5: Show by induction that $1^2 + 2^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$.

Solution: Our claim is that $\sum_{j=1}^{n} j^2 = n(n + 1)(2n + 1)/6$.

- **Base case:** This is true for $n = 1$ since $1^2 = 1(1 + 1)(2 \cdot 1 + 1)/6$.
- **Inductive step:** For $n \geq 2$ we can assume that it's true for $1, \ldots, n - 1$. Now, we have $1^2 + \ldots + n^2 = (1^2 + \ldots + (n - 1)^2) + n^2$. For the first part of the sum in parentheses, we have by the inductive step that it equals $(n - 1)n(2(n - 1) + 1)/6$. We then have the extra n^2 term. Adding $(n - 1)n(2(n - 1) + 1)/6 + n^2$ gives $n(n + 1)(2n + 1)/6$.

Exercise 6: A robot starts off in an infinite grid of cells, at the location $(0, 0)$. At each time step he can move diagonally to the topleft, topright, bottomleft, or bottomright (see the picture below).



Can the robot ever reach the cell (0,1)? Either show a way he can, or show that he can't using induction.

Solution: The robot cannot ever reach (0,1). This is because of the robot's coordinates are (x, y), then we always have that $x + y$ is even.

We prove the following claim inductively: for all points in time n, the robot's location (x, y) at time n satisfies that $x + y$ is even.

- **Base case:** This is true at time step 0 since his location is $(0, 0)$ which has $0 + 0 = 0$ even.
- **Inductive step:** For $n \geq 1$ we can assume that it's true for $0, \ldots, n - 1$. Now, suppose his location at time step $n - 1$ is (x, y). We assume $x + y$ is even. Then in the next step his location will either be $(x - 1, y - 1)$, $(x + 1, y - 1)$, $(x + 1, y + 1)$, or $(x + 1, y - 1)$. In any case, the sum of his new coordinates will either be $x + y$, $x + y - 2$, or $x + y + 2$. All of these are even since $x + y$ is even.