Addis Ababa University, Amist Kilo July 11, 2011
Algorithms and Programming for High Schoolers

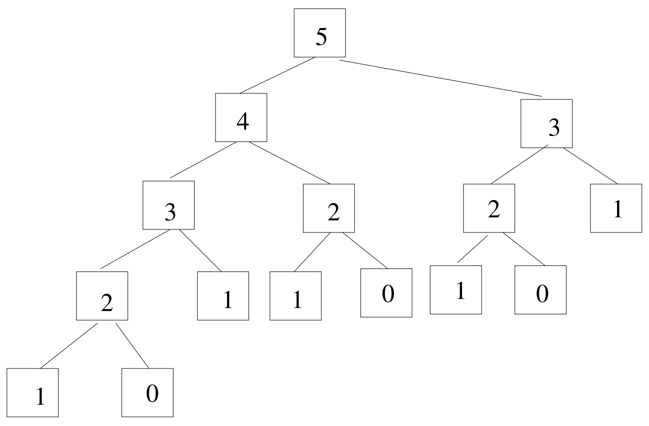
Lecture 6

Memoization:

Memoization is recursion, but where you remember answers that you have computed before. For many problems, this allows trading a small amount of extra memory usage (to store previously seen answers) to have significant gains in running time. Recall our code from lecture 3 to compute the nth Fibonacci number.

```
def fibonacci(n):
    if n<2:
        return 1
    return fibonacci(n-1) + fibonacci(n-2)
```

The recursion tree for computing `fibonacci`(5) looks as follows.



Note there is quite a bit of wasted work in this computation. `fibonacci`(3) is computed twice throughout the tree, and `fibonacci`(2) is computed three times. When computing `fibonacci`(n) for larger n, in fact many values are repeatedly computed an exponentially large number of times (recall that in Lab 5 we showed that the code above takes $O(((1+\sqrt{5})/2)^n) \approx O(1.618^n)$ time to compute `fibonacci`(n)).

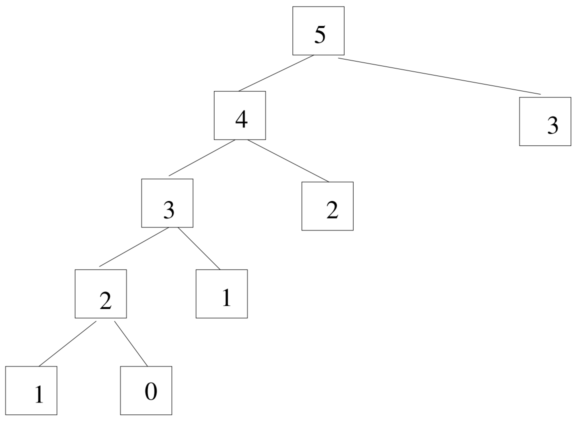
Now here's an implementation of `fibonacci` using memoization.

```
def memoizedFibonacci(n, memory):
    if n<2:
        return 1
    if memory[n] != -1:
        return memory[n]
    memory[n] = memoizedFibonacci(n-1, memory) + memoizedFibonacci(n-2,memory)
    return memory[n]

def fibonacci(n):
    # memory[i] contains the answer we've already computed for
    # fibonacci(i), where it's -1 if we haven't computed it yet
    memory = []
    for i in xrange(n+1):
        memory += [-1]
    return memoizedFibonacci(n, memory)
```

The above code takes only $O(n)$ time to compute `fibonacci`(n) using just $O(n)$ additional memory. There is a big difference between $O(n)$ time and $O(1.618^n)$ time for even fairly small n. For example, when $n = 50$, 1.618^n is almost thirty billion.

Here's what the recursion tree looks like for the new memoized implementation.



For general n, the tree just looks like a long path going down the left side, with one node hanging off the right side of every node. In general, there are less than $2n$ calls to the `fibonacci` function throughout the recursion tree.

Another good example of dynamic programming is making change. In the US coin system, and also formerly in Ethiopia, there were coins for 1 cent, 5 cents, 10 cents, 25 cents, and 50 cents. Suppose someone gives you an amount and asks you to make change, how do you do it? For example, what if they want you to make change for 88 cents? You could give them 88 1-cent pieces, but that would be annoying. You could also give 17 5-cent pieces and 3 1-cent pieces, but that is still kind of annoying. Instead, usually we keep giving the largest coin we can until it's too big, then move to the next largest coin, etc. For example, with 88 cents we would first give a 50-cent piece, then a 25, then a 10, then three 3's, thus altogether giving 6 coins. In fact for the value 88 cents, making change in 6 coins is the best possible, and in fact this approach of always giving the biggest coin first will always make change using the least number of coins.

However, if we switch to another country which might have a different coin system, this approach isn't always the best possible. For example, consider a country that has coins with the following values: 1, 4, 5, and 10 cents. Then, the strategy of giving the highest value coin first would give 5+1+1+1 (4 coins) to make change for 8 cents, whereas it's possible to do it with 2 coins (4+4).

Suppose we want to implement a function `makeChange`(n, L) as follows. n is some value we want to make change for, in cents, and L is a list of values of all the different types of coins that exist in our country. The return value of `makeChange` should be the minimum number of coins needed to make change for n cents. We can implement it recursively.

```
# if we can't make change for n cents using L, returns -2
def makeChange(n, L):
    if n==0:
        return 0
    # if possible to make change for n, the answer is definitely less
    # than n + 1
    answer = n + 1
    for x in L:
        if x <= n:
            val = makeChange(n-x, L)
            if val != -2:
                answer = min(answer, 1 + val)
    if answer == n + 1:
        answer = -2
    return answer
```

The above code though runs quite slowly. In fact, it's not too hard to convince yourself that you can make it take exponential time in the value n. You can speed it up, however, with memoization.

```
def memoizedMakeChange(n, L, memory):
    if n==0:
        return 0
    if memory[n] != -1:
        return memory[n]
    # if possible to make change for n, the answer is definitely less
    # than n + 1
    answer = n + 1
    for x in L:
        if x <= n:
            val = memoizedMakeChange(n-x, L, memory)
            if val != -2:
                answer = min(answer, 1 + val)
    if answer == n + 1:
        answer = -2
    memory[n] = answer
    return memory[n]

# if we can't make change for n cents using L, returns -2
def makeChange(n, L):
    # memory[i] contains the answer we've already computed to
    # make change for i cents, where it's -1 if we haven't computed it
    # yet. if we can't make change for n cents, we put -2 as the
    # value.
    memory = []
    for i in xrange(n+1):
        memory += [-1]
    return memoizedMakeChange(n, L, memory)
```

The above memoized version only takes $O(nm)$ time to figure out the optimal number of coins to make change for n cents, where m is the size of the list L.